I am indebted to Céline Lacaux for pointing out some misleading ways of using little os notation to prove the two results below, as published in the original version of the article. Here are updated, cleaner proofs using domination arguments.

Updated proof of Proposition 5

Proof of Proposition 5. By conditioning and using the independence of $X := f(A_0)$ and $X_j := f(A_j)$ for $j \geq 1$, and that of K_A and X_j, $j \geq 1$, we obtain

$$\begin{aligned}\mathbb{P}\left(H^R > x\right) &= 1 - \sum_{k=0}^{\infty} \mathbb{P}\big(X \leq x \mid K_A = k\big)\big(\mathbb{P}\left(X \leq x\right)\big)^k \mathbb{P}\left(K = k\right) \\ &= 1 - \sum_{k=0}^{\infty} \mathbb{P}\big(X \leq x \mid K_A = k\big)(1 - \mathbb{P}\left(X > x\right))^k \mathbb{P}\left(K = k\right).\end{aligned}$$

Trivially, $1 = \mathbb{P}\left(X > x\right) + \mathbb{P}\left(X \leq x\right)$, it follows that

$$\begin{aligned}1 &- \sum_{k=0}^{\infty} \mathbb{P}\big(X \leq x \mid K_A = k\big)(1 - \mathbb{P}\left(X > x\right))^k \mathbb{P}\left(K = k\right) \\ &= \mathbb{P}\left(X > x\right) + \sum_{k=0}^{\infty} \mathbb{P}\left(X \leq x, K_A = k\right)\left(1 - (1 - \mathbb{P}\left(X > x\right))^k\right).\end{aligned}$$

We first derive a lower bound on the above quantity. Using the basic inequality $1 - (1 - \mathbb{P}\left(X > x\right))^k \geq k\,\mathbb{P}\left(X > x\right)$, it follows that

$$\begin{aligned}\mathbb{P}\left(H^R > x\right) &\geq \mathbb{P}\left(X > x\right) + \mathbb{P}\left(X > x\right)\Big(\mathbb{E}\left[K_A\right] - \sum_{k=0}^{\infty} k\mathbb{P}\left(X > x, K_A = k\right)\Big) \\ &\geq \mathbb{P}\left(X > x\right)\Big(1 + \mathbb{E}\left[K_A\right] - \mathbb{E}\left[K_A \mathbf{1}_{\{X>x\}}\right]\Big).\end{aligned}$$

By regular variation of (X, K_A) with index $\alpha > 1$, K_A is integrable and, hence, $\mathbb{E}\left[K_A \mathbf{1}_{\{X>x\}}\right] = o(1)$, as $x \to \infty$. Overall, this yields as a lower bound

$$\mathbb{P}\left(H^R > x\right) = \mathbb{P}\left(X > x\right)\big(1 + \mathbb{E}\left[K_A\right] + o(1)\big), \text{ as } x \to \infty.$$

For an upper bound, we write $\big(1 - \mathbb{P}\left(X > x\right)\big) = \exp\big(k\log(1 - \mathbb{P}\left(X > x\right))\big)$ and we use a Taylor expansion on the log term yielding

$$\exp\big(k\log(1 - \mathbb{P}\left(X > x\right))\big) = \exp\big(-k\mathbb{P}\left(X > x\right) - k o(\mathbb{P}\left(X > x\right))\big), \text{ as } x \to \infty,$$

and recalling the basic inequality $1 - e^{-x} \leq x$, an upper bound is given by

$$\begin{aligned}\mathbb{P}\left(H^R > x\right) &= \mathbb{P}\left(X > x\right) + \sum_{k=0}^{\infty} \mathbb{P}\left(X \leq x, K_A = k\right)\left(1 - \exp(k \log(1 - \mathbb{P}\left(X > x\right)))\right) \\ &\leq \mathbb{P}\left(X > x\right)\left(1 + \sum_{k=0}^{\infty} k\,\mathbb{P}\left(X \leq x, K_A = k\right) + \mathcal{O}(1) \sum_{k=0}^{\infty} k\,\mathbb{P}\left(X \leq x, K_A = k\right)\right) \\ &\leq \mathbb{P}\left(X > x\right)\left(1 + \mathbb{E}\left[K_A\right] + \mathcal{O}(1)\right), \text{ as } x \to \infty.\end{aligned}$$

Combining with the above upper bound, the desired result follows. □

Updated proof of Proposition 7

Proof. Proof of Proposition 7 By conditioning and using the independence of a generic $X := f(A_0)$ and H_j^H (the subcluster maximum being independent from X), and that of L_A and H_j^H, for $j \geq 1$, we obtain as in the proof of Proposition 5

$$\mathbb{P}\left(H^H > x\right) = 1 - \sum_{k=0}^{\infty} \mathbb{P}\big(X \leq x \mid L_A = k\big)\big(1 - \mathbb{P}\left(H^H > x\right)\big)^k \mathbb{P}\left(L_A = k\right).$$

From here on, the proof follows the same lines as that of Proposition 5, except that the (generic) tail of H^H appears here rather than the tail of X. The proof is omitted for brevity, but we retrieve

$$P(H^H > x) = \mathbb{P}\left(X > x\right)\left(\frac{1}{1 - \mathbb{E}\left[L_A\right]} + \mathcal{O}(1)\right), \text{ as } x \to \infty,$$

which yields the desired result. □